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A Natural Interpolation Formula for Cauchy-Type Singular Integral Equations with Generalized Kernels

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A natural interpolation formula, appropriate for a class of Cauchy-type singular integral equations with generalized kernels, is suggested. This formula is a generalization of the corresponding formula for the same class of equations, but with regular kernels, which has been already proved equivalent (under mild assumptions) to Nyström's natural interpolation formula for Fredholm integral equations of the second kind. The special case when the Gauss–Jacobi quadrature rule is used for the numerical solution of Cauchy-type singular integral equations with generalized kernels is considered in detail. The superiority of the suggested natural interpolation formula to the Lagrangian interpolation formula is illustrated in a numerical application.

1. INTRODUCTION

The numerical solution of Fredholm integral equations of the second kind (along a finite real interval) by the quadrature (Nyström) method is the classical method for the solution of these equations [1, 2]. In this method, use is made of Nyström's natural interpolation formula for the estimation of the unknown function along the whole integration interval on the basis of its values at the nodes of the quadrature rule used. The numerical solution of Cauchy-type singular integral equations (also along a finite real interval) by the quadrature method appeared with the work of Kalandiya (see, e.g., [3]) thirty years later and was further developed by several authors (see, e.g., [4–11] and the literature reported there). As regards the natural interpolation formula for Cauchy-type singular integral equations, it was suggested by this author and reported for the first time in [9]. For Cauchy-type singular integral equations reducible to Fredholm integral equations of the second kind, this natural interpolation formula was proved (under appropriate but mild assumptions) completely equivalent to Nyström's natural interpolation formula [10–11]. The usefulness of the natural interpolation formula for the numerical solution of Cauchy-type singular integral equations (clear from the numerical results of [9]) and the fact that it is analogous to the classical Nyström's natural interpolation formula are indications that this formula will be widely used in the future.

Here we shall generalize the results of [9–11] to the case of Cauchy-type singular

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integral equations with generalized kernels, that is, kernels presenting a pole at an end-point of the integration interval. We shall use as a model a simple integral equation of this kind, but the present results can also be generalized to apply to more complicated cases. The case when the Gauss–Jacobi quadrature rule is used will be considered in detail and the results of [9] will be derived as a very special case. A numerical application, based on this quadrature rule, will also be made and the superiority of the natural interpolation formula to the Lagrangian (polynomial) interpolation formula will be seen in this application.

Finally, it should be emphasized that Cauchy-type singular integral equations with generalized kernels are not reducible to ordinary Fredholm integral equations of the second kind, in contrast to what happens with Cauchy-type singular integral equations with regular kernels. Hence, any attempt to generalize the results of [10–11] in order to show the equivalence of the suggested natural interpolation formula to Nyström's seems simply meaningless.

2. The Natural Interpolation Formula

Let us consider the simple Cauchy-type singular integral equation with a generalized kernel

$$\int_{-1}^{1} \left[\frac{1}{t-x} + \frac{\lambda}{t+x-2} + k(t, x)\right] \varphi(t)\, dt = f(x), \qquad -1 < x < 1, \tag{1}$$

where λ is a known constant and $0 < |\lambda| < 1$, $\varphi(t)$ is the unknown function, $k(t, x)$ is a known regular kernel, and $f(x)$ is a known regular function. Generally, the condition

$$\int_{-1}^{1} \varphi(t)\, dt = 0 \tag{2}$$

supplements (1) in order that (1) possess a unique solution. Singular integral equation (1) in the special case when $k(t, x) \equiv 0$ appears in at least three branches of physics and engineering (waveguide theory [12], dislocations in metallurgy [13], and elasticity [14]); moreover, it possesses a closed-form solution (see, e.g., [15] and the corresponding literature, including [12, 13]).

The difficulty of (1) is due to the appearance of the term $\lambda/(t + x - 2)$ in its kernel, which tends to infinity as $t, x \to 1$ simultaneously. Clearly, (1) is the simplest singular integral equation of the class considered in this paper (with generalized kernels). Much more complicated equations of this kind appear in many problems of physics and engineering and have been solved numerically by several researchers.

Following the developments of [14], we take into account the behavior of $\varphi(t)$ as $t \to \pm 1$ by means of the weight function

$$w(t) = (1-t)^{\alpha}(1+t)^{\beta}, \qquad -1 < \alpha < 0, \quad \beta = -\tfrac{1}{2}, \tag{3}$$

with

$$\cos \pi\alpha = -\lambda, \qquad -1 < \alpha < 0, \tag{4}$$

that is,

$$\varphi(t) = w(t)\, g(t), \tag{5}$$

where $g(t)$ is a new unknown function. Next, we use an appropriate quadrature rule for regular integrals

$$\int_{-1}^{1} w(t)\, g(t)\, dt \simeq \sum_{i=1}^{n} A_i\, g(t_i) \tag{6}$$

(where t_i are the nodes and A_i the weights, both dependent on n), convergent for increasing values of n for $g \in \mathscr{C}[-1, 1]$. For Cauchy-type principal value integrals, (6) takes the form [16]

$$\int_{-1}^{1} w(t) \frac{g(t)}{t-x}\, dt \simeq \sum_{i=1}^{n} A_i \frac{g(t_i)}{t_i - x} - K_n(x)\, g(x), \qquad -1 < x < 1, \quad x \neq t_i, \quad i = 1,..., n, \tag{7}$$

where

$$K_n(x) = q_n(x)/\sigma_n(x) \tag{8}$$

with

$$\sigma_n(x) = \prod_{i=1}^{n} (x - t_i), \tag{9}$$

$$q_n(x) = \int_{-1}^{1} \frac{\sigma_n(t)}{x - t}\, dt. \tag{10}$$

This is exactly quadrature rule (6) with the principal part of the error term (due to the pole of the integrand) taken into account.

From the results of [17] it follows that (7) converges as $n \to \infty$ for $g \in \mathscr{C}^1[-1, 1]$ if (6) converges as $n \to \infty$ for $g \in \mathscr{C}[-1, 1]$. Finally, from (6) we also obtain

$$\int_{-1}^{1} w(t) \frac{g(t)}{t + x - 2}\, dt \simeq \sum_{i=1}^{n} A_i \frac{g(t_i)}{t_i + x - 2} - K_n(2 - x)\, g(2 - x), \qquad -1 < x < 1, \tag{11}$$

where the last term in (11) is the principal part of the error term [18], generally tending to infinity as $x \to 1 + 0$. Clearly, (11) converges for the same class of functions $g(x)$ for which (6) converges.

By using (6), (7), and (11) in (1) and (2), we rewrite these equations as [6],

$$\sum_{i=1}^{n} A_i \left[\frac{1}{t_i - x} + \frac{\lambda}{t_i + x - 2} + k(t_i, x)\right] g_n(t_i) - F_n(x)\, g_n(x) = f(x), \tag{12}$$

$$\sum_{i=1}^{n} A_i\, g_n(t_i) = 0, \tag{13}$$

where $g_n(x)$ is an approximation to $g(x)$ and

$$F_n(x) = K_n(x) + \lambda K_n(2 - x). \tag{14}$$

In (12) we have assumed that [6]

$$g(2 - x) = g(x), \tag{15}$$

extending the definition of $g(x)$ in the interval $(1, 3]$. This is reasonable if $g'(1) = 0$ and it happens if $\lambda \in (0, 1)$ in (1) [19]. If $\lambda \in (-1, 0)$, we have to take into account the results of [19] and introduce a new unknown function $\tilde{g}(t)$, such that $\tilde{g}'(1) = 0$, to substitute for $g(t)$. Now, by applying (12) at $n - 1$ collocation points x_k, the roots of $F_n(x)$, the existence of which was proved in [4], Eqs. (12) and (13) reduce to a system of n linear algebraic equations, from which $g_n(t_i)$ are determined [6, 19].

At this point we can make the following comments: In most cases in practice, the second term of the right-hand side of (11) is ignored, but (11) still remains convergent (for a fixed value of x). This term is in reality the principal part of the corresponding error term. The reason we do not ignore this term during the numerical solution of singular integral equations is simply that $K_n(2 - x) \to \infty$ as $x \to 1 + 0$ (for a constant value of n) although $K_n(2 - x) \to 0$ as $n \to \infty$ (for a constant value of x) for all classical quadrature rules. During the numerical solution of singular integral equations we use collocation points x_k very close to 1 as n increases and this frequently causes an increase of the absolute value of $K_n(2 - x)$ for the first collocation point (near the point $x = 1$). To be sure about the accuracy of our numerical results, it is convenient to take into account the aforementioned term in (11). But since we do not know the exact value of $g(2 - x)$ and since we are interested in this value only when $x \to 1$, we can use $g(1)$ instead of $g(2 - x)$, provided, of course, that $g'(1) = 0$. Then $g(x)$ does not tend to infinity as $x \to 1$. In practice it is more convenient to use (15) for the extension of the definition of $g(x)$ for $x > 1$, but very near to 1. In either case, we succeed in taking into account the main part of the error term in (11), expressed by the second term of its right-hand side. The error made by replacing the value of $g(2 - x)$ by the value of $g(1)$ (or, better, by the value of $g(x)$) is of the order of $1 - x$ (if $g'(1) = 0$) and is insignificant compared to the principal part of the error term in (11). In any case, we can add that in many numerical results concerning the solution of singular integral equations with generalized kernels, the second term in the right-hand side of (11) was ignored, in contrast to what is done here as well as in [6].

Now, as regards the determination of $g_n(x)$ along the whole interval $[-1, 1]$, we suggest, generalizing the results of [9–11], that the following natural interpolation formula be used (resulting directly from (12))

$$g_n(x) = -\frac{1}{F_n(x)}\left\{f(x) - \sum_{i=1}^{n} A_i\left[\frac{1}{t_i - x} + \frac{\lambda}{t_i + x - 2} + k(t_i, x)\right] g_n(t_i)\right\},$$

$$x \neq t_i, \quad i = 1,\ldots, n, \quad x \neq x_k, \quad k = 1,\ldots, (n-1). \tag{16}$$

Clearly, this formula is based on the values $g_n(t_i)$ of $g_n(x)$ at the nodes t_i of the quadrature rule (6). As regards the values of $g_n(x)$ at the collocation points x_k, we apply L'Hôpital's rule to (16) or, equivalently, we differentiate (12) with respect to x and apply the resulting equation for $x = x_k$, taking also into account the fact that the collocation points x_k are the roots of $F_n(x)$. Then we obtain

$$g_n(x_k) = -\frac{1}{F_n'(x_k)}\left\{f'(x_k) - \sum_{i=1}^{n} A_i\left[\frac{1}{(t_i - x_k)^2} - \frac{\lambda}{(t_i + x_k - 2)^2} + \frac{\partial k}{\partial x}(t_i, x_k)\right] g_n(t_i)\right\}, \quad k = 1,\ldots, (n-1). \tag{17}$$

Equations (16) and (17) permit the determination of $g_n(x)$ along the whole interval $[-1, 1]$ on the basis of its values at the nodes t_i. We can add that when $x \to t_i$, we obtain from (16)

$$\lim_{x \to t_i} g_n(x) = \lim_{x \to t_1} [(\sigma_n(x)/q_n(x))(A_i/(t_i - x))\, g_n(t_i)], \qquad i = 1,\ldots, n, \tag{18}$$

(where (14) was also taken into consideration). By applying L'Hôpital's rule to (18), we obtain

$$\lim_{x \to t_i} g_n(x) = -(\sigma_n'(t_i)/q_n(t_i))\, A_i\, g_n(t_i), \qquad i = 1,\ldots, n. \tag{19}$$

But we know that (see, e.g., [18])

$$A_i = -q_n(t_i)/\sigma_n'(t_i), \qquad i = 1,\ldots, n, \tag{20}$$

for the weights of an ordinary interpolatory quadrature rule of the form (6). Hence, (19) reduces to

$$\lim_{x \to t_i} g_n(x) = g_n(t_i), \qquad i = 1,\ldots, n, \tag{21}$$

an expected result. Equations (17) and (21) both resulted from (16), our basic natural interpolation formula, by elementary limiting procedures; they are not independent of (16).

We shall now consider the behavior of $F_n(x)$ in (16) as $x \to 1$ (the singular point of (1)). We rewrite (10) as

$$q_n(z) = \int_{-1}^{1} w(t) \frac{\sigma_n(t) - \sigma_n(z)}{t - z} \, dt + \sigma_n(z) \int_{-1}^{1} \frac{w(t)}{t - z} \, dt, \qquad z = x + iy \notin [-1, 1]. \tag{22}$$

The integrand $[\sigma_n(t) - \sigma_n(z)]/(t - z)$ in the first term of the right-hand side of (22) is clearly a polynomial (because of (9)). Moreover, because of well-known properties of Cauchy-type integrals [20] and expression (3) for the weight function $w(t)$, we see directly that

$$q_n(z) \sim A(z-1)^\alpha + B, \qquad z \notin [-1, 1], \tag{23}$$

(where A and B are constants) as $z \to 1$. For $z \in [-1, 1]$ we obtain from (23) on the basis of the second Plemelj formula [20]

$$\begin{aligned} q_n(z) &\sim \tfrac{1}{2} A(1-z)^\alpha [\exp(i\pi\alpha) + \exp(-i\pi\alpha)] + B \\ &= A(1-z)^\alpha \cos \pi\alpha + B, \qquad z \in (-1, 1). \end{aligned} \tag{24}$$

Since $\sigma_n(z)$ is well behaved as $z \to 1$, because of its definition (9), it follows from (14) that $F_n(z)$ is also well behaved as $z \to 1$ because of (23), (24), and the determination of the order of the singularity α from (3) and (4). Hence, the denominator $F_n(x)$ in (16) tends to a finite value (and different from zero) as $x \to 1$. The same happens—and it is easier to see this—as $x \to -1$. Hence, (16) can be used at the end-points of the integration interval $[-1, 1]$. This is necessary if these points are not included among the nodes t_i of quadrature rule (6), as is usually the case.

3. Application of the Gauss–Jacobi Quadrature Rule

The well-known Gauss–Jacobi quadrature rule [18], associated with a weight function $w(t)$ of form (3), is the most frequently used as the quadrature rule (6). For this rule we have

$$\sigma_n(z) = P_n^{(\alpha,\beta)}(z), \tag{25}$$

where $P_n^{(\alpha,\beta)}(z)$ is the Jacobi polynomial of degree n associated with $w(t)$, and

$$q_n(z) = \Pi_n^{(\alpha,\beta)}(z), \tag{26}$$

where [18]

$$\Pi_n^{(\alpha,\beta)}(z) = \int_{-1}^{1} w(t) \frac{P_n^{(\alpha,\beta)}(t)}{z - t} \, dt, \tag{27}$$

a formula analogous to (10). Moreover, for $z \notin [-1, 1]$ [18]

$$\Pi_n^{(\alpha,\beta)}(z) = 2(z-1)^\alpha (z+1)^\beta Q_n^{(\alpha,\beta)}(z), \tag{28}$$

where $Q_n^{(\alpha,\beta)}(z)$ is the Jacobi function of the second kind [21, p. 170].

We take further into account that [21, p. 171]

$$\begin{aligned} Q_n^{(\alpha,\beta)}(z) = &-\frac{\pi}{2 \sin \pi\alpha} P_n^{(\alpha,\beta)}(z) + 2^{\alpha+\beta-1} \frac{\Gamma(\alpha)\, \Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)} (z-1)^{-\alpha}(z+1)^{-\beta} \\ &\times F(n+1, -n-\alpha-\beta; 1-\alpha; \tfrac{1}{2}-(z/2)), \qquad z \notin [-1, 1], \end{aligned} \tag{29}$$

where $\Gamma(x)$ denotes the gamma function and $F(a, b; c; z)$ the ordinary hypergeometric function. By combining (28) and (29) and taking into account that [21, p. 169]

$$P_n^{(\alpha,\beta)}(1) = \Gamma(n+\alpha+1)/\Gamma(n+1)\, \Gamma(\alpha+1), \tag{30}$$

we see directly that

$$\frac{\Pi_n^{(\alpha,\beta)}(z)}{P_n^{(\alpha,\beta)}(z)} \sim -2^\beta \frac{\pi}{\sin \pi\alpha} (z-1)^\alpha + 2^{\alpha+\beta} \frac{\Gamma^2(\alpha+1)\, \Gamma(n+1)\, \Gamma(n+\beta+1)}{\alpha\Gamma(n+\alpha+1)\, \Gamma(n+\alpha+\beta+1)} \tag{31}$$

as $z \to 1$ ($z \notin [-1, 1]$). In this way, the constants A and B in (23) can be explicitly determined in the case of the Gauss–Jacobi quadrature rule.

As regards $F_n(x)$, it is determined from (14), which, because of (8), (25), and (26), is written as

$$F_n(x) = \frac{\Pi_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(x)} + \lambda \frac{\Pi_n^{(\alpha,\beta)}(2-x)}{P_n^{(\alpha,\beta)}(2-x)}, \qquad x \in (-1, 1). \tag{32}$$

The value of $F_n(x)$ for $x \to 1$ is of particular importance since this is necessary in the natural interpolation formula (16) for the determination of $g_n(1)$ (if 1 is not a node in quadrature rule (6)). This value is the reduced stress intensity factor in crack problems in plane elasticity. Now, because of (4), (23), (24), and (31), we obtain from (32)

$$F_n(1) = 2^{\alpha+\beta} \frac{1-\cos \pi\alpha}{\alpha} \frac{\Gamma^2(\alpha+1)\, \Gamma(n+1)\, \Gamma(n+\beta+1)}{\Gamma(n+\alpha+1)\, \Gamma(n+\alpha+\beta+1)}. \tag{33}$$

By inserting this value in (16), we find

$$\begin{aligned} g_n(1) = &-2^{-\alpha-\beta} \frac{\alpha}{1-\cos \pi\alpha} \frac{\Gamma(n+\alpha+1)\, \Gamma(n+\alpha+\beta+1)}{\Gamma^2(\alpha+1)\, \Gamma(n+1)\, \Gamma(n+\beta+1)} \\ &\times \left\{ f(1) - \sum_{i=1}^{n} A_i \left[\frac{1}{t_i - 1} + \frac{\lambda}{t_i + 1} + k(t_i, 1) \right] g_n(t_i) \right\}. \end{aligned} \tag{34}$$

A very special case of (1) occurs when $\lambda = 0$. Then we obtain from (4): $\alpha = -\frac{1}{2}$. In this case, (1) is an ordinary Cauchy-type singular integral equation. Moreover, in this case (34) reduces to

$$g_n(1) = \frac{1}{n}\left\{\frac{1}{\pi} f(1) - \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{t_i - 1} + k(t_i, 1)\right] g_n(t_i)\right\}, \tag{35}$$

since $\Gamma(\frac{1}{2}) = \pi^{1/2}$ and $A_i = \pi/n$ in this special case, where the Gauss–Jacobi quadrature rule reduces to the Gauss–Chebyshev quadrature rule. Equation (35) coincides with the corresponding formula found in [9] in this special case (with a slightly different notation and $k(t, x) \equiv 0$).

4. A Numerical Application

We apply the previous developments to the classical singular integral equation [6, 12–15, 19]

$$\int_0^1 t^{-\gamma}(1-t)^{-\delta}\left(\frac{1}{t-x} + \frac{\lambda}{t+x}\right) g(t)\, dt = 1, \tag{36}$$

accompanied by the condition

$$\int_0^1 t^{-\gamma}(1-t)^{-\delta} g(t)\, dt = 0, \tag{37}$$

which assures the uniqueness of its solution. The exponents γ and δ are determined from

$$\cos \pi\gamma = -\lambda, \qquad 0 < \gamma < 1, \quad \delta = \tfrac{1}{2}. \tag{38}$$

These equations are analogous to (3) and (4), but the integration interval is now the interval $[0, 1]$ instead of $[-1, 1]$.

Now we take into account the closed-form solution of (36) and (37) [13]

$$\begin{aligned} g(t) = (t^{\gamma}/\pi \sin(\pi\gamma/2)) \{&\gamma(1+t)^{-1/2} \cosh[\gamma \cosh^{-1}(1/t)] \\ &- (1-t)^{1/2} \sinh[\gamma \cosh^{-1}(1/t)]\}. \end{aligned} \tag{39}$$

From (39) we obtain directly the value of $g(0)$

$$g(0) = 2^{\gamma}(\gamma - 1)/2\pi \sin(\pi\gamma/2). \tag{40}$$

As regards the numerical solution of (36) and (37), it was considered in a series of papers (see, e.g., [6, 14, 19]).

Here we wish to show the effectiveness of the natural interpolation formula proposed previously for singular integral equations with generalized kernels, exactly as we have done in [9] for the corresponding interpolation formula for ordinary singular integral equations (reducible to Fredholm integral equations of the second kind). Since the worst point of the integration interval is the "singular" point $t = 0$, we present our numerical results for this point. Moreover, since the aim of this paper is to show the efficiency of the natural interpolation formula (compared to the Lagrangian interpolation formula), we assumed that these formulas were based on exact values for the unknown function $g(t)$ at the nodes used, which are the same for both these formulas.

As regards the natural interpolation formula, we have used (34), which in our case takes the following form:

$$g_n(1) = -\frac{\gamma\Gamma(n-\gamma+1)\,\Gamma(n-\gamma-\delta+1)}{\Gamma^2(-\gamma+1)\,\Gamma(n+1)\,\Gamma(n-\delta+1)}\left[\frac{1}{1+\lambda} - \sum_{i=1}^{n} \frac{A_i}{t_i} g_n(t_i)\right]. \tag{41}$$

In this equation, the values of the nodes t_i (roots of the shifted Jacobi polynomial $\tilde{P}_n^{(-\gamma,-\delta)}(t)$) and of the corresponding weights A_i were obtained from [22]. The values of $g_n(t_i)$ were calculated by using (39) as already mentioned. As regards the Lagrangian (polynomial) interpolation formula, it can be easily constructed on the basis of the values of $g_n(t_i)$, either directly as an interpolation polynomial or by taking into account the results of Krenk [23], based on the orthogonality properties of the Jacobi polynomials.

In Table I we display the numerical results for $g_n(0)$ in the case when $\gamma = 0.7$ (so that $\lambda > 0$, (38), as already mentioned; in fact, $\lambda = 0.587785$ in this case) and for $n = 1, \dots, 6$ by the aforementioned interpolation formulas, denoted by the subscripts (N, natural) and (L, Lagrangian), respectively. We present also the values of $g(t)$ at the node t_1 nearest to the point $t = 0$, the theoretical value of $g(0)$, determined from (40), as well as the relative errors for both interpolation formulas. It is clear from the results of Table I that the natural interpolation formula for singular integral equations with generalized kernels is superior to the corresponding Lagrangian interpolation formula. Therefore, it is believed that the natural interpolation formula may substitute

TABLE I

Comparison of the Values of $g(0)$ (and the Corresponding Relative Errors) of the Solution $g(t)$ of (36) and (37) (for $\gamma = 0.7$ and $n = 1, \dots, 6$), Obtained by the Natural Interpolation Formula (Subscript N) and by the Lagrangian Interpolation Formula (Subscript L)

n	$g(t_1)$	$g_N(0)$	$g_L(0)$	$\varepsilon_N(0)$	$\varepsilon_L(0)$
1	−0.015072	−0.074961	−0.015072	−13.89%	−82.69%
2	−0.075586	−0.085831	−0.100111	−1.40%	15.00%
3	−0.083046	−0.086693	−0.089800	−0.41%	3.16%
4	−0.085106	−0.086897	−0.088015	−0.18%	1.11%
5	−0.085925	−0.086971	−0.087504	−0.09%	0.52%
6	−0.086325	−0.087004	−0.087305	−0.06%	0.29%

Theoretical value: −0.087053

in practical applications of singular integral equations with generalized kernels for the Lagrangian interpolation formula, exactly as has been the case with Fredholm integral equations of the second kind [1, 2].

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